J11 BA

Answers pps 19-27 Selected Problems

■ p. 19

[5.1] Substitute b for c in $\log_a b = \frac{\log_c b}{\log_c a}$, to obtain $\log_a b = \frac{1}{\log_b a}$

[5.2] $\log_a b \cdot \log_b c \, \log_c a = \log_a b \, \frac{\log_a c}{\log_a b} \, \frac{\log_a a}{\log_a c} = 1$

■ p. 21

[8] Let $n =$ number of panes. Then $a_n =$ the intensity of the light after passing through n panes. So, $a_n = a_0\left(\frac{9}{10}\right)^n$. When $a_n = \frac{1}{3} a_0$,

$$\frac{1}{3} a_0 = a_0\left(\frac{9}{10}\right)^n$$

$$\Longleftrightarrow \log \frac{1}{3} = n \log \frac{9}{10}$$

$$\Longleftrightarrow n(2 \log 3 - 1) = -\log 3$$

$$\Longleftrightarrow n(\log 9 - \log 10) = -\log 3$$

$$\Longleftrightarrow n = \frac{-0.4771}{2\,(0.4771) - 1}$$

$$\Longleftrightarrow n = \frac{-\log 3}{2 \log 3 - 1}$$

$$\Longleftrightarrow n = 10.41$$

Therefore, 11 panes of glass will reduce the intensity to no more than $\frac{1}{3}$ of its original intensity.

[9] Let n represent the number of 20 minute periods. Then a_n = the number of bacteria present after n such periods. When there are one million bacteria, $a_n = 1000000 = 10^6 = 20\,(2^n)$.

$$10^6 = 20 \cdot 2^{\mathrm{n}} \Longleftrightarrow 10^5 = 2^{\mathrm{n}+1} \Longleftrightarrow \log 10^5 = (\mathrm{n}+1)\log 2 \Longleftrightarrow \mathrm{n} = \frac{5}{\log 2} - 1 \Longleftrightarrow \mathrm{n} = 15.61.$$

15.61 periods of 20 minutes is 312.2 minutes. Therefore, after 312.2 minutes, there will be one million bacteria.

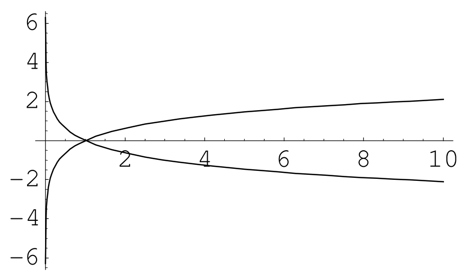
■ p. 23

[2.1] $1 < \log x < 2$ [2.2] $-2 < \log x \le 0$ [2.3] $-3 \le \log x \le 0$ [2.4] $n \le \log x < n+1$

[3] Let $x = 3^{50}$. Then $\log x = \log 3^{50} = 50 \log 3 \Longleftrightarrow \log x = 50\,(0.4771) = 23.855$. So, $23 < \log x < 24$. This means $10^{23} < x < 10^{24}$. Therefore, 3^{50} has 24 digits.

■ p. 24

[3.1] $\log_3 \frac{1}{x} = -\log_3 x$. Since $(x,\ y) \in \log_3 x \Longleftrightarrow (x,\ -y) \in \log_3 \frac{1}{x}$, The graphs are symmetric with respect to the x-axis.



[3.2] $\log_{\frac{1}{3}} x = \frac{\log_3 x}{\log_3(\frac{1}{3})} = -\log_3 x$. Since $(x,\ y) \in \log_3 x \Longleftrightarrow (x,\ -y) \in \log_{\frac{1}{3}} x$, The graphs are symmetric with respect to the x-axis.

[3.3] The graphs are identical, because $\log_{\frac{1}{3}} \frac{1}{x} = -\log_3 \frac{1}{x} = \log_3 x$.

[4.1] $3 \log_4 3 = 3\, \frac{\log_2 3}{\log_2 4} = \frac{3}{2} \log_2 3 < 2 \log_2 3$. So $3 \log_4 3 < 2 \log_2 3$.

[4.2] Notice that $\log_3 2 = \log_3 \sqrt[3]{8}$ and $\frac{2}{3} = \log_3 3^{\frac{2}{3}} = \log_3 \sqrt[3]{9}$. Now, $\log_3 x$ is an increasing function of x for all $x \in \mathbb{R}$. Since $\sqrt[3]{9} > \sqrt[3]{8}$, $\log_3 3^{\frac{2}{3}} > \log_3 2$. Therefore $\frac{2}{3} > \log_3 2$.

[5] Let $x = 2^{30},\ y = 3^{20}$. Then $\log x = 30 \log 2$, $\log y = 20 \log 3$.
$\frac{\log x}{\log y} = \frac{30 \log 2}{20 \log 3} = \frac{3\,(0.3010)}{2\,(.4771)} = 0.946342 < 1$. So, $\log x < \log y$. Since the function $\log_{10}$ is strictly increasing on $\mathbb{R}$, conclude that $x < y$; that is, $2^{30} < 3^{20}$.

[6.1] Let $y = 4^{15}$. Then $\log y = 15 \log 4 = 30 \log 2 = 30\,(0.3010) = 9.03$. Since $9 < \log y < 10,\ 10^9 < y < 10^{10}$. Therefore, 4^{15} is a 10 digit number.

[6.2] Let $y = \left(\frac{1}{4}\right)^{20}$. Then $\log y = 20 \log \frac{1}{4} = -40 \log 2 = -40\,(0.3010) = -12.04$. Since $-13 < \log y < -12,\ 10^{-13} < y < 10^{-12}$. Therefore, the first non-zero digit appears in the 13 decimal place of the number $\left(\frac{1}{4}\right)^{20}$.

■ p. 25

[7.1] Solve $x = 3^{y-1}$ for y.
$x = 3^{y-1} \Longleftrightarrow \log x = (y-1) \log 3 \Longleftrightarrow \frac{\log x}{\log 3} + 1 = y \Longleftrightarrow y = \log_3 x + \log_3 3 \Longleftrightarrow y = \log_3(3\,x)$.

The question does not ask you to then prove that your result is the inverse of $y = 3^{x-1}$, but you should be able to do so. The proof would include the following.

$3^{x-1} \circ \log_3(3\,x)$

$= \log_3(3\,(3^{x-1}))$

$= \log_3 3 + (x-1) \log_3 3$

$= 1 + (x-1)$

$= x$

$= I(x)$

This shows that the composition $3^{x-1} \circ \log_3(3\,x)$ is the identity function.

[7.2] Solve for y, $x = 2\log_4 y$. $x = 2\log_4 y \Longleftrightarrow x = \log_4 y^2 \Longleftrightarrow 4^x = y^2 \Longrightarrow y = \pm\sqrt{4^x}$, note that $\sqrt{4^x} > 0$, for all $x \in \mathbb{R}$. This means $-\sqrt{4^x}$ is negative for all $x \in \mathbb{R}$. The argument of $y = 2\log_4 x$ must be a positive number, so $-\sqrt{4^x}$ cannot be an argument. Consequently, the inverse of $y = 2\log_4 x$ is $y = \sqrt{4^x}$.

[8.1] $\log_{a^2} b^2 = \frac{\log_a b^2}{\log_a a^2} = \frac{2\log_a b}{2\log_a a} = \log_a b$.

[8.2] $\log_{a^2} b^3 = \frac{\log_a b^3}{\log_a a^3} = \frac{3\log_a b}{3\log_a a} = \log_a b$.

■ p. 27

[5.1] Suppose a,b,c all positive and $a < b$. Then $\frac{b}{a} > 1$. So, $\left(\frac{b}{a}\right)^c > 1$ and $\frac{b^c}{a^c} > 1$ which is equivalent to $a^c < b^c$.

[5.2] To see that this is not true for every a, b, $c > 0$, consider $c = \frac{1}{2}$, $a = 2$, $b = 3$. Then $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ and $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$. But $\frac{1}{8} < \frac{1}{4}$.

[6.1] Let $y = \left(\frac{1}{5}\right)^{15} = \left(\frac{2}{10}\right)^{15}$. Then $\log y = 15\log 2 - 15\log 10 = 15\,(.3010) - 15 = -10.485$. So, $-11 < \log\left(\frac{1}{5}\right)^{15} < -10 \Longleftrightarrow 10^{-11} < \left(\frac{1}{5}\right)^{15} < 10^{-10}$. The first non-zero digit is in the 11^{th} decimal place.

[6.2] Let $y = \left(\frac{2}{3}\right)^{10}$. Then $\log y = \log\left(\frac{2}{3}\right)^{10} = 10\log 2 - 10\log 3 = 10\,(.0301) - 10\,(.4771) = -4.47$. So, $-5 < \log\left(\frac{2}{3}\right)^{10} < -4 \Longleftrightarrow 10^{-5} < \left(\frac{2}{3}\right)^{10} < 10^{-4}$. The first non-zero digit is in the 5^{th} decimal place.

[7.1] $1.5^n > 100 \Longleftrightarrow \left(\frac{3}{2}\right)^n > 100 \Longleftrightarrow n\log\left(\frac{3}{2}\right) > \log 100 \Longleftrightarrow n(\log 3 - \log 2) > 2 \Longleftrightarrow n > \frac{2}{\log 3 - \log 2} =$

$$\frac{2}{0.4771 - 0.3010} = 11.35$$

So, if $n \in \mathbb{Z}$, then $n = 12$. [WHY does $a > b$ guarantee that $\log a > \log b$?]

[7.2] $0.4^n > 0.001 \Longleftrightarrow \left(\frac{4}{10}\right)^n > 10^{-3}$

$$\Longleftrightarrow n\log\left(\frac{4}{10}\right) > -3$$

$$\Longleftrightarrow n(\log 4 - \log 10) > -3$$

$$\Longleftrightarrow n > \frac{-3}{2\log 2 - 1} = \frac{-3}{2\,(.3010) - 1} = 7.537$$

So $n > 8$.

[8.1] First note that since $\log_3(x-3)$ is undefined for $x \leq 3$. So we require $x > 3$.
$\log_3(x-3) \leq 3 \Longleftrightarrow 3^3 \leq x-3 \Longleftrightarrow 27 \leq x-3 \Longleftrightarrow 24 \leq x$. Therefore, $3 < x \leq 30$.

[8.2] First note that since $\log_{0.5}(2\,x)$ is undefined for $x \leq 0$. So we require $x > 0$.
$\log_{0.5}(2\,x) \leq 2 \Longleftrightarrow \left(\frac{1}{2}\right)^2 \leq 2\,x \Longleftrightarrow 2^{-2} \leq 2\,x \Longleftrightarrow 2^{-3} \leq x$. Therefore, $0 < x \leq \frac{1}{8}$.